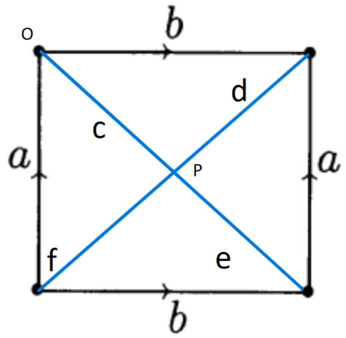
TOPOLOGY - III, SOLUTION SHEET 9

Exercise 1. We refer to parts of of Hatcher's book for this exercise.

(1) Please refer to example 0.3 on page 6.
(2) Please refer to example 0.1 on page 6.
(3) Please refer to the first example on page 5.
(4) Please refer to the discussion at the end of page 51.
(5) Please refer to example 0.4 on page 6.
(6) Please refer to example 0.6 on page 6.

Exercise 2. We denote $\mathbb{Z}/2\mathbb{Z}$ by F and compute the simplicial homology of T^2 in F- coefficients.



Consider the delta complex structure of T^2 coming fro gluing four $2-$ simplices as prescribed by the above diagram. We pressume the orientation of the blue line segments to be towards the centre as in the last exercise of sheet 3. We therefore have 4 2-simplices, 6 1-simplices namely a,b,...,f and 2 0-simplices given by O and P in the delta complex structure. We obtain the following cellular chain complex:

$$0 \to F^4 \xrightarrow{\alpha} F^6 \xrightarrow{\beta} F^2 \to 0.$$

With the usual boundary maps. We have that $\beta(a) = O + O = 0, \beta(b) = O + O = 0$, and $\beta(c) = \beta(d) = \beta(e) = \beta(f) = O + P,$. Therefore $H_0(T^2; F) \cong F^2/ < O + P > \cong F$. It also follows that $Ker\, \beta$ has a $F-$ basis given by $c+d, d+e, e+f, a, b$. Where as image of α, which is the span of $a+c+f$, $a+d+e$, $b+c+d$, $b+f+e$ has the basis given by $a+c+f$, $a+d+e$, $b+c+d$. Hence $H_1(T^2; F) \cong F^2$ as it is the quotient of a five dimensional $F-$vector space by a three dimensional sub-space. Since the image of α is 3 dimensional, we obtain that $Ker\, \alpha = H_2(T^2; F) \cong F$ by the rank-nullity theorem of linear algebra.

We leave the computation of $H_*(\mathbb{RP}^2, F)$ to the reader since it is an easier version of the above computation and uses the same ideas.

Exercise 3. In the following solution all tensor product are over $\mathbb{Z}$. That is $\otimes := \otimes_{\mathbb{Z}}$. Recall the universal coefficient theorem, which says that for all abelian groups A, we have a split-exact sequence for all n:

$$0 \to H_n(X;\mathbb{Z}) \otimes A \to H_n(X;A) \to \mathrm{Tor}_1(H_{n-1}(X;\mathbb{Z}), A) \to 0.$$

(1) Recall that $\mathrm{Tor}_1(\mathbb{Z}^k, A) = 0$ for all indices k and abelian groups A. Hence we have that $H_k(S^n; G) \cong H_k(S^n) \otimes G$ is equal to G in degrees 0 and n and 0 otherwise.

(2) Since the homology groups of T^2 are all 0 or free abelian groups it follows from the argument above that $H_k(T^2; G) \cong H_k(T^2) \otimes G$ for all k and hence $H_0(T^2) = G$, $H_1(T^2) = G^2$ and $H_2(T^2) = G$.

(3) We make note of the following facts from algebra:
1. $\mathrm{Tor}_1(\mathbb{Q}, A) = 0$ for all abelian groups A.
2. $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} \cong \mathrm{Tor}_1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/(n,m)\mathbb{Z}$ for $n, m > 0$.
3. $T \otimes \mathbb{Q} = 0$ for any torsion Abelain group T.

By points 1 and 3 above, using the universal coefficient theorem and the homology of $\mathbb{RP}^2$ over $\mathbb{Z}$, we obtain $H_0(\mathbb{RP}^2; \mathbb{Q}) \cong \mathbb{Q}$ and $H_i(\mathbb{RP}^2; \mathbb{Q}) = 0$ for all $i > 0$. Using point 2 above, coupled with the universal coefficient theorem, we also obtain that $H_0(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, $H_1(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, $H_2(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. For $\mathbb{Z}/3\mathbb{Z}$ coefficients we have that $H_0(\mathbb{RP}^2; \mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$ and $H_i(\mathbb{RP}^2; \mathbb{Z}/3\mathbb{Z}) \cong 0$ for all $i > 0$.

Exercise 4. Throughout we will identify tuples (a, b, c, d) with the quaternion $a + bi + cj + dk$. Multiplication by i on the right is a linear, and hence continuous map. Indeed $(a+bi+cj+dk)\cdot i = -b+ai+dj-ck$. Moreover the vector (a,b,c,d) is normal to (-b,a,d,-c). Therefore $v \mapsto (v, vi)$ is a vector field on S^3. where by (v, vi) we mean the tangent vector vi at v. Similarly multiplication by j,k define vector fields. Since for all $v \in S^3$, the tangent vectors vi, vj and vk are orthonormal, they are linearly independent.